Module-2: Simultaneous Differential Equations of First-Order and First Degree

1. Introduction

Systems of simultaneous differential equations of the first-order and first-degree of the type

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n, t), \quad (i = 1, 2, \cdots, n) \tag{1}$$

arise frequently in mathematical physics, e.g., in the general theory of radioactive transformation, harmonic oscillator, heavy string hanging from two points of support etc. The problem is to find n functions x_i which depend on t and the initial condition and which satisfy the set of equations (1) identically in t.

For example, an nth-order differential equations given by

$$\frac{d^n x}{dt^n} = f\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots, \frac{d^{n-1}x}{dt^{n-1}}, t\right)$$

may be written in the form

$$\frac{dx}{dt} = y_1, \quad \frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = y_3, \cdots, \frac{dy_{n-1}}{dt^{n-1}} = f(x, y_1, y_2, \cdots, y_{n-1}, t) \tag{2}$$

This shows that it is a special case of (1).

2. Simultaneous Equations in Three Variables

Let us consider equations in three variables x, y, z as

$$P_1 dx + Q_1 dy + R_1 dz = 0,$$
$$P_2 dx + Q_2 dy + R_2 dz = 0,$$

where each of P_i, Q_i, R_i, $(i = 1, 2)$, is a function of x, y, z. It follows that

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

$$\text{i.e.} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{3}$$

where $P = Q_1R_2 - Q_2R_1$, $Q = R_1P_2 - R_2P_1$, $R = P_1Q_2 - P_2Q_1$ and each of P, Q, R is a function of x, y, z.

Equations (3) over a set of simultaneous equations of first order and first degree. The existence and uniqueness of solutions of equations (3) follows from the following theorem which we state without proof.

Theorem 1: *Suppose the functions $f(x,y,z)$ and $g(x,y,z)$ be defined and continuous in the region $|x-a| < k$, $|y-b| < l$, $|z-c| < m$, where a, b, c and k, l, m are constants and satisfy the Lipschitz condition*

$$\begin{aligned} |f(x,y,z) - f(x,\eta,\zeta)| &< A_1|y-\eta| + B_1|z-\zeta|, \\ |g(x,y,z) - g(x,\eta,\zeta)| &< A_2|y-\eta| + B_2|z-\zeta|, \end{aligned} \tag{4}$$

in the defined region, in which A_i B_i, $(i = 1,2)$, are some finite constants. Then there exists a unique pair of function $y(x)$ and $z(x)$, continuous and having continuous derivatives in a suitable interval $|x-a| < h$, which satisfy the differential equations

$$\frac{dy}{dx} = f(x,y,z), \quad \frac{dz}{dx} = g(x,y,z), \tag{5}$$

and the conditions $y(a) = b$, $z(a) = c$.

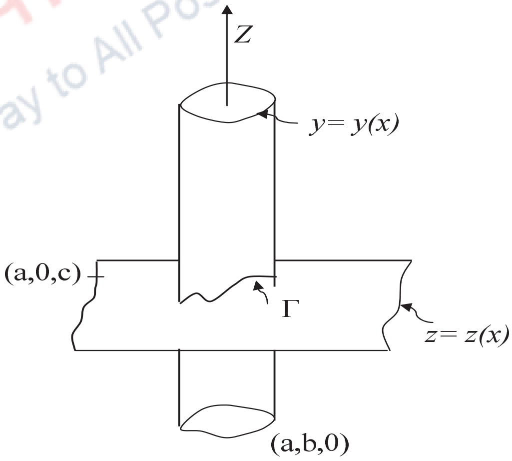

Fig. 1

The above theorem shows that there exist two cylinders $y = y(x)$ and $z = z(x)$ passing through the points $(a,b,0)$ and $(a,0,c)$ such that the equations (5) hold. The complete solutions of these pair of equations, therefore, consist of the set of points lying on both the cylinders, i.e., on the curve of intersection Γ passing through the point (a,b,c). Since a, b, c are arbitrary, so the general solution of the pair of equations consists of curves of intersection of one-parameter system of cylinders of which $y = y(x)$

and $z = z(x)$ are also two members. Hence, the general solution of a set of simultaneous equations of the type (3) is a two-parameter family of curves in three-dimensional space.

3. Methods of Solution

To solve equations of the form (3), it may be noted that if one can derive solutions of these equations of the form

$$u_1(x,y,z) = c_1, \quad u_2(x,y,z) = c_2, \tag{6}$$

where c_1 and c_2 are arbitrary constants, then by varying these constants, one can obtain a two-parameter family of curves satisfying the differential equations (3).

Let us now discuss some methods for solving equations of the type (3).

I. Method-1:

Consider the equation $u_1(x,y,z) = c_1$ to be a one-parameter system of surfaces so that any tangential direction through the point (x,y,z) to this surface satisfies the relation

$$\frac{\partial u_1}{\partial x}dx + \frac{\partial u_1}{\partial y}dy + \frac{\partial u_1}{\partial z}dz = 0.$$

To determine u_1, we look for three functions $P'(x,y,z)$, $Q'(x,y,z))$ and $R'(x,y,z)$ such that

$$PP' + QQ' + RR' = 0 \tag{7}$$

where $P' = \frac{\partial u_1}{\partial x}$, $Q' = \frac{\partial u_1}{\partial y}$, $R' = \frac{\partial u_1}{\partial z}$ and $P'dx + Q'dy + R'dz$ is an exact differential.

Similarly, we find u_2.

Example 1: Find the integral curves of the equations

$$\frac{dx}{xz - y} = \frac{dy}{yz - x} = \frac{dz}{1 - z^2}$$

Solution: Here $P = xz - y$, $Q = yz - x$, $R = 1 - z^2$.

Let us choose $P' = z + 1$, $Q' = z + 1$, $R' = x + y$. Then

$PP' + QQ' + RR' = (xz - y)(z + 1) + (yz - x)(z + 1) + (1 - z^2)(x + y) = 0$

and $P'dx + Q'dy + R'dz = (z+1)dx + (z+1)dy + (x+y)dz = d\{(x+y)(z+1)\}$, an exact differential. Hence $u_1 = (x+y)(z+1)$.
Again, we choose $P'' = z-1$, $Q'' = 1-z$, $R'' = x-y$ so that $PP''+QQ''+RR'' = (xz-1)(z-1)+(yz-x)(1-z)+(1-z^2)(x-y) = 0$
and $P''dx + Q''dy + R''dz = (z-1)dx + (1-z)dy + (x-y)dz = d\{(x-y)(z-1)\}$, an exact differential so that $u_2 = (x-y)(z-1)$.

Hence, the integral curves of the given differential equations are the members of the two-parameter family

$$(x+y)(z+1) = c_1, \text{ and } (x-y)(z-1) = c_2,$$

where c_1 and c_2 are arbitrary constants.

II. Method-2:

Let us choose two sets of three functions, (P', Q', R') and (P'', Q'', R'') such that each of

$$\frac{P'dx + Q'dy + R'dz}{PP' + QQ' + RR'} \text{ and } \frac{P''dx + Q''dy + R''dz}{PP'' + QQ'' + RR''}$$

is an exact differential dw' and dw'' respectively, say. Since, each of these is equal to $\frac{dx}{P}$, so there must exist the relation $dw' = dw''$ i.e., $w' = w'' + c$, between x, y and z, c being an arbitrary constant.

Example 2: Find the integral curves of the equations

$$\frac{dx}{y+\alpha z} = \frac{dy}{z+\beta x} = \frac{dz}{x+\gamma y},$$

α, β, γ being constants.

Solution: If λ, μ, ν are constant multipliers, then each of the given ratios is equal to

$$\frac{\lambda dx + \mu dy + \nu dz}{\lambda(y+\alpha z) + \mu(z+\beta x) + \nu(x+\gamma y)}$$

and this expression will be an exact differential, provided it is of the form $\frac{1}{\rho}\frac{\lambda dx+\mu dy+\nu dz}{\lambda x+\mu y+\nu z}$ and this is possible iff

$$\begin{aligned} -\rho\lambda + \beta\mu + \nu &= 0, \\ \lambda - \rho\mu + \gamma\nu &= 0 \\ \alpha\lambda + \mu - \rho\nu &= 0. \end{aligned} \tag{8}$$

This possesses a solution if ρ is a root of the equation

$$\begin{vmatrix} -\rho & \beta & 1 \\ 1 & -\rho & \gamma \\ \alpha & 1 & -\rho \end{vmatrix} = 0, \text{ i.e. of } \quad \rho^3 - (\alpha+\beta+\gamma)\rho - (1+\alpha\beta\gamma) = 0$$

Let ρ_1, ρ_2, ρ_3 be the roots of this equation. Substituting the values of ρ_i in (8) and solving for λ_i, μ_i, ν_i, $(i = 1,2,3)$, we get the expressions

$$\frac{1}{\rho_1}\frac{\lambda_1 dx + \mu_1 dy + \nu_1 dz}{\lambda_1 x + \mu_1 y + \nu_1 z}, \qquad \frac{1}{\rho_2}\frac{\lambda_2 dx + \mu_2 dy + \nu_2 dz}{\lambda_2 x + \mu_2 y + \nu_2 z} \qquad \frac{1}{\rho_3}\frac{\lambda_3 dx + \mu_3 dy + \nu_3 dz}{\lambda_3 x + \mu_3 y + \nu_3 z}$$

$$\text{Taking } dw' = \frac{1}{\rho_1}\frac{\lambda_1 dx + \mu_1 dy + \nu_1 dz}{\lambda_1 x + \mu_1 y + \nu_1 z}, \quad \text{and } dw'' = \frac{1}{\rho_2}\frac{\lambda_2 dx + \mu_2 dy + \nu_2 dz}{\lambda_2 x + \mu_2 y + \nu_2 z}$$

the equation $dw' = dw''$ gives on integration $\lambda_1 x + \mu_1 y + \nu_1 z = c_1(\lambda_2 x + \mu_2 y + \nu_2 z)^{\rho_1/\rho_2}$

If we take $dw'' = \frac{1}{\rho_3}\frac{\lambda_3 dx+\mu_3 dy+\nu_3 dz}{\lambda_3 x+\mu_3 y+\nu_3 z}$, then $\lambda_1 x + \mu_1 y + \nu_1 z = c_2(\lambda_3 x + \mu_3 y + \nu_3 z)^{\rho_1/\rho_3}$

Thus, the required solutions are

$$\lambda_1 x + \mu_1 y + \nu_1 z = c_1(\lambda_2 x + \mu_2 y + \nu_2 z)^{\rho_1/\rho_2}, \quad \lambda_1 x + \mu_1 y + \nu_1 z = c_2(\lambda_3 x + \mu_3 y + \nu_3 z)^{\rho_1/\rho_3}$$

III. Method-3:

If one of the variables, say z, is absent from one equation of the set (3) then the integral curves can be obtained in a simple way. Thus, when z is absent in P and Q, then we have $\frac{dx}{P} = \frac{dy}{Q}$ i.e. $\frac{dy}{dx} = \frac{Q}{P}$, $(p \neq 0)$, which has a solution of the type $f(x,y,c_1) = 0$. Elimination of x or y from one or other equations of (3), we obtain another relation between x and z or y and z from which we get the second solution.

Example 3: Solve the equations

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}.$$

Solution: From the last two equations, we get

$$\frac{dy}{dz} = \frac{y}{z} \Rightarrow y = c_1 z, \quad c_1 \text{ being constant.}$$

Also, from the first and last equations we have

$$\frac{dx}{(c_1^2+1)z^2 - x^2} = \frac{dz}{-2xz} \Rightarrow (c_1^2+1)dz = -\frac{2xzdx - x^2dz}{z^2} = -d\left(\frac{x^2}{z}\right)$$

$$\text{Integrating, we get } \quad (c_1^2+1)z = -\frac{x^2}{z} + c_2 \Rightarrow c_1^2 z^2 + z^2 = -x^2 + c_2 z$$

$$\text{or } \; y^2 + z^2 = -x^2 + c_2 z, \text{ i.e. } \; x^2+y^2+z^2 = c_2 z$$

Hence, the required solutions are $y = c_1 z$, $x^2+y^2+z^2 = c_2 z$.

4. Orthogonal Trajectories of a System of Curves on a Surface

Suppose a system of curves lie on the surface whose equation is

$$F(x, y, z) = 0 \tag{9}$$

Then a system of curves cutting every curve of the system lying on the surface at right angles is called the *system of orthogonal trajectories* on the surface of the given system of curves. The original system of curves may, therefore, be thought as the intersection of the surface (9) with the one-parameter family of surfaces

$$G(x, y, z) = C, \tag{10}$$

where C is a parameter.

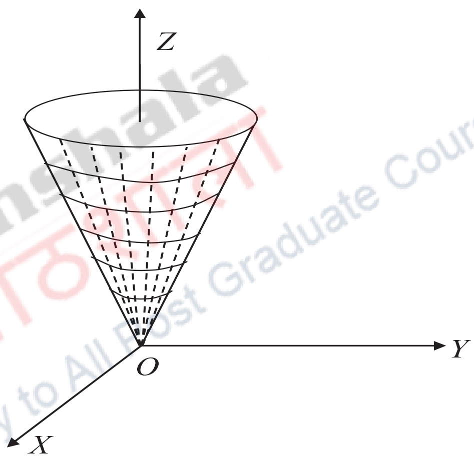

Fig. 2

As an illustration, consider the system of circles (shown by full lines in Fig. 2) lying on the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ by the system of parallel planes $z = C$, where C is a parameter. Then the generators (shown by dotted lines in Fig. 2) are orthogonal trajectories.

In general, the tangential direction (dx, dy, dz) to the given curve through the point (x, y, z) on the surface (9) satisfies the equations

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

and $$\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz = 0$$

so that $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{11}$$

where $$P = \frac{\partial(F, G)}{\partial(y, z)}, \quad Q = \frac{\partial(F, G)}{\partial(z,\ x)}, \quad R = \frac{\partial(F, G)}{\partial(x, y)}$$

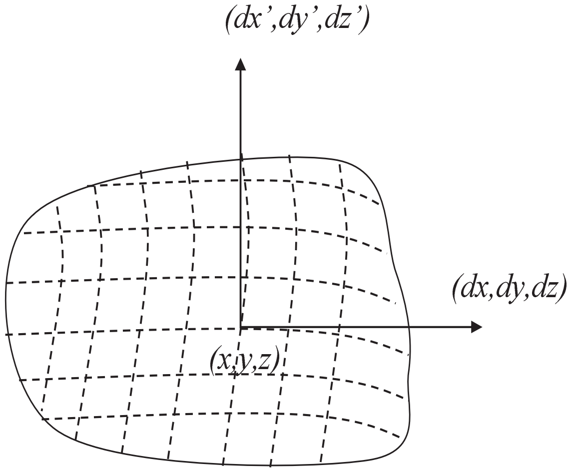

Fig. 3

Now the curve through the point (x, y, z) of the orthogonal system has tangential direction (dx', dy', dz') (Fig. 3) lying on the surface (9) implying

$$\frac{\partial F}{\partial x}dx' + \frac{\partial F}{\partial y}dy' + \frac{\partial F}{\partial z}dz' = 0 \quad (12)$$

and is perpendicular to the original system of curves. So, we have from (12)

$$Pdx' + Qdy' + Rdz' = 0 \quad (13)$$

Equations (12) and (13) together yield

$$\frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'} \quad (14)$$

where $$P' = R\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial z}, \quad Q' = P\frac{\partial F}{\partial z} - R\frac{\partial F}{\partial z}, \quad R' = Q\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial y}. \quad (15)$$

The solution of the equation (14) with the relation (9) gives the system of orthogonal trajectories.

Example 4: Find the orthogonal trajectories on the surface $x^2 + y^2 + 2fyz + d = 0$ of its curves of intersection with planes parallel to the plane xoy.

Solution: Let $z = k$, (k being constant), be the plane parallel to the $x \cdot y$. Then the given system of conics is characterized by pair of equations

$$xdx + (y + fz)dy + fydz = 0 \text{ and } dz = 0.$$

which are equivalent to

$$\frac{dx}{y + fz} = \frac{dy}{-x} = \frac{dz}{0}$$

The system of orthogonal trajectories is, therefore, given by the pair of equations

$$xdx + (y + fz)dy + fydz = 0 \text{ and } (y + fz)dx - xdy = 0$$

i.e. by $$\frac{dx}{fxy} = \frac{dy}{fy(y + fz)} = \frac{dz}{-x^2 - (y + fz)^2}$$

i.e. $$\frac{dx}{fxy} = \frac{dy}{fy(y + fz)} = \frac{dz}{d - f^2z^2} \quad (\because x^2 + y^2 + 2fz + d = 0 \Rightarrow x^2 + (y + fz)^2 = f^2z^2 - d)$$

$$\Rightarrow \frac{dx}{fxy} = \frac{fzdy}{f^2yz(y + fz)} = \frac{fydz}{fy(d - f^2z^2)} = \frac{d(fyz)}{fy(fyz + d)} \Rightarrow \frac{dx}{x} = \frac{d(fyz + d)}{fyz + d}$$

$$\Rightarrow fyz + d = cx, \quad c \text{ being a parameter.}$$

Thus the orthogonal trajectories are $fyz + d = cx$, $x^2 + y^2 + 2fyz + d = 0$.